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Galilean invariance of subgrid-scale stress models in the large-eddy simulation of turbulence

By CHARLES G. SPEZIALE†
Princeton University, Princeton, NJ 08544

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The modelling of the subgrid-scale stresses in the large-eddy simulation of turbulence is examined from a theoretical standpoint. While there are a variety of approaches that have been proposed, it is demonstrated that one of the more recent models gives rise to equations of motion for the large eddies of turbulence which are not Galilean-invariant. Consequently, this model cannot be of any general applicability, since it is inconsistent with the basic physics of the problem, which requires that the description of the turbulence be the same in all inertial frames of reference. Alternative models that have been proposed which are properly invariant are discussed and compared.

1. Introduction

During the past decade a considerable amount of research has been conducted on the large-eddy simulation of turbulence. In these simulations, the small-scale turbulence (which is nearly isotropic) is modelled by an eddy-viscosity type of approach while the large-scale structures are calculated directly (cf. Deardorff 1970; Clark, Ferziger & Reynolds 1979; Moin & Kim 1982). The flow-field variables are averaged spatially with a filter function that constitutes a Dirac delta sequence as first proposed by Reynolds (1895). In this manner, the high-frequency Fourier components of the velocity in space are filtered out and the flow properties become more regular.

The purpose of the present paper is to examine in more detail various models that have been proposed for the subgrid-scale stresses. In one such model, which has received much attention during the past few years (see Biringen & Reynolds 1981; Moin & Kim 1982), the Leonard stresses are calculated directly while the subgrid-scale cross-stresses and subgrid-scale Reynolds stresses are approximated by utilizing the Smagorinsky model, which is essentially an eddy-viscosity approach. It will be proven that the subgrid-scale cross-stresses are not Galilean-invariant, while the Smagorinsky model is, and hence this approach gives rise to equations of motion for the large eddies that are *not* Galilean invariant. Since the Navier–Stokes equations as well as their filtered form (which after modelling yield the equations of motion for the large eddies) do exhibit this invariance, it is clear that this model is deficient. To be more specific, this model is inconsistent with the basic physics of the problem, which requires that the description of the turbulence be the same in all inertial frames of reference. In order to avoid this problem, all subgrid-scale stress models must be form-invariant under a Galilean transformation (hence terms like the subgrid-scale cross-stresses which are not Galilean invariant must be modelled with terms that are equivalently

† Permanent address: Stevens Institute of Technology, Hoboken, NJ 07030.

not invariant so that the model retains the same form in all inertial frames of reference). It will be demonstrated that by making a small modification in the linear combination model of Bardina, Ferziger & Reynolds (1983) (the Bardina constant must be adjusted from a value of 1.1 to 1) the physical constraint of Galilean invariance can be satisfied identically. In addition, it will be shown that the older models of Deardorff (1970) and Clark *et al.* (1979) give rise to equations of motion for the large eddies that are also Galilean-invariant. However, these models are inferior to the modified linear-combination model as a result of unnecessary inaccuracies introduced in the modelling of the Leonard stresses (the Leonard stresses can be calculated directly). Additional physical constraints that subgrid-scale stress models should be subject to will be discussed briefly in §3 along with the prospects for future research.

2. Models for the subgrid-scale stresses

In the large-eddy simulation of turbulence, any flow variable ϕ is decomposed into a mean and fluctuating part respectively as follows:

$$\phi = \overline{\phi} + \phi', \tag{1}$$

where

$$\overline{\phi} = \int_D G(\boldsymbol{x}-\boldsymbol{x}', \Delta)\,\phi(\boldsymbol{x}')\,\mathrm{d}^3x'. \tag{2}$$

In (2), G is a filter function which depends on the relative position vector $\boldsymbol{x}-\boldsymbol{x}'$ in the fluid domain D and on the computational mesh size Δ. The function G is normalized, i.e.

$$\int_D G(\boldsymbol{x}-\boldsymbol{x}', \Delta)\,\mathrm{d}^3x' = 1, \tag{3}$$

and is usually taken to be a Gaussian distribution in an infinite flow domain or a piecewise continuous distribution of bounded support otherwise (cf. Deardorff 1970; Leonard 1974). In the limit as Δ approaches zero, (2) becomes a Dirac delta sequence (cf. Arfken 1970), i.e.

$$\lim_{\Delta\to 0}\int_D G(\boldsymbol{x}-\boldsymbol{x}', \Delta)\,\phi(\boldsymbol{x}')\,\mathrm{d}^3x' \equiv \int_D \delta(\boldsymbol{x}-\boldsymbol{x}')\,\phi(\boldsymbol{x}')\,\mathrm{d}^3x' = \phi(\boldsymbol{x}), \tag{4}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}')$ is the Dirac delta function. As a direct result of the Riemann–Lebesgue theorem, (2) substantially reduces the amplitude of the high-frequency Fourier components in space of any flow variable ϕ. Consequently, $\overline{\phi}$ can be more accurately termed the large-scale component of ϕ and ϕ' the residual or small-scale field. It should be noted at this point that, unlike in the more traditional Reynolds averaging,

$$\overline{\overline{\phi}} \neq \overline{\phi}, \tag{5}$$

$$\overline{\phi'} \neq 0 \tag{6}$$

in general.

We will consider the turbulent flow of a homogeneous and incompressible viscous fluid which is governed by the Navier–Stokes equations

$$\frac{\partial u_k}{\partial t} + u_l\frac{\partial u_k}{\partial x_l} = -\frac{\partial p}{\partial x_k} + \nu\,\nabla^2 u_k, \tag{7}$$

where $\boldsymbol{u}$ is the velocity field, p is the modified pressure, which includes the gravitational body-force potential, and ν is the kinematic viscosity of the fluid. Equation (7) is solved subject to the continuity equation

$$\nabla\cdot\boldsymbol{u} = 0. \tag{8}$$

By filtering the Navier–Stokes equations (7) and continuity equation (8), we obtain the equations of motion for the large-scale eddies which are given by (cf. Deardorff 1970)

$$\frac{\partial \bar{u}_k}{\partial t}+\bar{u}_l\frac{\partial \bar{u}_k}{\partial x_l} = -\frac{\partial \bar{p}}{\partial x_k}+\nu\nabla^2\bar{u}_k-\frac{\partial \tau_{kl}}{\partial x_l}, \tag{9}$$

$$\nabla\cdot\bar{\boldsymbol{u}} = 0, \tag{10}$$

where
$$\tau_{kl} = \overline{\bar{u}_k\bar{u}_l}-\bar{u}_k\bar{u}_l+\overline{u'_k\bar{u}_l}+\overline{\bar{u}_k u'_l}+\overline{u'_k u'_l} \tag{11}$$

are the subgrid-scale stresses. Here

$$L_{kl} = \overline{\bar{u}_k\bar{u}_l}-\bar{u}_k\bar{u}_l, \tag{12}$$

$$C_{kl} = \overline{u'_k\bar{u}_l}+\overline{\bar{u}_k u'_l}, \tag{13}$$

$$R_{kl} = \overline{u'_k u'_l}, \tag{14}$$

are respectively referred to as the Leonard stresses, the subgrid-scale cross-stresses and the subgrid-scale Reynolds stresses. In deriving (9) and (10), the commutative property of the filtering process with time and space derivatives has been utilized, i.e. for any flow variable ϕ

$$\overline{\frac{\partial \phi}{\partial t}} = \frac{\partial \bar{\phi}}{\partial t}, \tag{15}$$

$$\overline{\frac{\partial \phi}{\partial x_k}} = \frac{\partial \bar{\phi}}{\partial x_k}. \tag{16}$$

It should be noted that, for (16) to hold with full rigour, the filter function G must vanish at the boundary of its support. Equations (9) and (10) are not closed because of the presence of additional unknown terms in $\boldsymbol{\tau}$. Closure is usually achieved by taking $\boldsymbol{\tau}$ to be some functional† of the global filtered velocity $\bar{\boldsymbol{u}}$, i.e.

$$\tau_{kl}(\boldsymbol{x},t) = \tau_{kl}[\bar{\boldsymbol{u}}(\boldsymbol{x}',t);\boldsymbol{x}], \quad \boldsymbol{x}'\in D. \tag{17}$$

Specific forms of (17) that have been studied in the literature will be discussed later.

Now we will examine the invariance of the equations of motion (7)–(10) under the Galilean group of transformations, which are given by

$$\boldsymbol{x}^* = \boldsymbol{x}+\boldsymbol{V}t+\boldsymbol{b}, \quad t^* = t \tag{18}$$

where $\boldsymbol{V}$ and $\boldsymbol{b}$ are any constant vectors. The Galilean group of transformations yield frames of reference whose motions differ by a constant translational velocity. Hence, if $\boldsymbol{x}$ constitutes an inertial frame of reference, then $\boldsymbol{x}^*$ will represent the class of inertial frames of reference. By differentiating (18), it is obvious that

$$\boldsymbol{u}^* = \boldsymbol{u}+\boldsymbol{V}, \quad \frac{\partial}{\partial x_k^*} = \frac{\partial}{\partial x_k}, \quad \frac{\partial}{\partial t^*} = \frac{\partial}{\partial t}-V_k\frac{\partial}{\partial x_k} \tag{19}$$

† It should be noted here that the Leonard stress $\boldsymbol{L}$ can be calculated directly so it is actually only necessary to provide closure models for $\boldsymbol{C}$ and $\boldsymbol{R}$ – a fact which has been made use of in the more recent models.

under the Galilean group of transformations. The direct substitution of (19) in (7) and (8) yields the equations

$$\frac{\partial u_k^*}{\partial t^*}+u_l^*\frac{\partial u_k^*}{\partial x_l^*}=-\frac{\partial p^*}{\partial x_k^*}+\nu\nabla^{*2}u_k^*, \tag{20}$$

$$\nabla^*\cdot \boldsymbol{u}^*=0, \tag{21}$$

and hence the Navier–Stokes equations are Galilean-invariant – a fact that has long been known. In deriving (20) we have, of course, made use of the fact that $p^*=p$ under the Galilean group (this results from the fact that the concept of force is frame-independent). It will now be demonstrated that the filtered form of the Navier–Stokes equations are also Galilean-invariant. Since the Navier–Stokes equations themselves are Galilean-invariant, in order to accomplish this task it is only necessary to show that the filtered part of a Galilean-invariant function is also Galilean-invariant. Given that

$$\phi^*=\phi \tag{22}$$

under the Galilean group of transformations (18), then

$$\bar{\phi}^*=\int_D G(\boldsymbol{x}^*-\boldsymbol{x}'^*)\,\phi^*(\boldsymbol{x}'^*)\,\mathrm{d}^3x'^*. \tag{23}$$

However, from (18) it is clear that

$$\boldsymbol{x}^*-\boldsymbol{x}'^*=\boldsymbol{x}+\boldsymbol{V}t+\boldsymbol{b}-(\boldsymbol{x}'+\boldsymbol{V}t+\boldsymbol{b})=\boldsymbol{x}-\boldsymbol{x}', \tag{24}$$

$$\mathrm{d}^3x'^*=\left|\frac{\partial x_k'^*}{\partial x_l'}\right|\mathrm{d}^3x'=\mathrm{d}^3x', \tag{25}$$

where $|\cdot|$ denotes the determinant. The direct substitution of (22), (24) and (25) into (23) yields the result

$$\bar{\phi}^*=\int_D G(\boldsymbol{x}-\boldsymbol{x}')\,\phi(\boldsymbol{x}')\,\mathrm{d}^3x'=\bar{\phi}, \tag{26}$$

which completes the proof. Since (9) and (10) are obtained simply by filtering the Navier–Stokes equations, as a result of (26) we have

$$\frac{\partial \bar{u}_k^*}{\partial t^*}+\bar{u}_l^*\frac{\partial \bar{u}_k^*}{\partial x_l^*}=-\frac{\partial \bar{p}^*}{\partial x_k^*}+\nu\nabla^{*2}\bar{u}_k^*-\frac{\partial \tau_{kl}^*}{\partial x_l^*}, \tag{27}$$

$$\nabla^*\cdot\bar{\boldsymbol{u}}^*=0, \tag{28}$$

where
$$\tau_{kl}^*=\overline{\bar{u}_k^*\bar{u}_l^*}-\bar{u}_k^*\bar{u}_l^*+\overline{u_k'^*\bar{u}_l^*}+\overline{\bar{u}_k^*u_l'^*}+\overline{u_k'^*u_l'^*},$$

and thus the equations of motion for the large eddies are Galilean-invariant, as would be expected on physical grounds. Consequently, the basic physics of the problem requires that the description of the turbulence as a whole as well as the description of the evolution of the large scales of turbulence (or any subset of scales for that matter) *be the same in all inertial frames of reference.*

In the large-eddy simulations of turbulence by Biringen & Reynolds (1981) and Moin & Kim (1982), the equations of motion for the large eddies given by (9) and (10) are solved in the equivalent form†

$$\frac{\partial \bar{u}_k}{\partial t}+\overline{\bar{u}_l\frac{\partial \bar{u}_k}{\partial x_l}}=-\frac{\partial \bar{P}}{\partial x_k}+\nu\nabla^2\bar{u}_k-\frac{\partial T_{kl}}{\partial x_l}, \tag{29}$$

$$\nabla\cdot\bar{\boldsymbol{u}}=0, \tag{30}$$

† The purpose of this modified approach was to make use of the fact that the Leonard stress can be calculated directly.

where
$$T_{kl} = Q_{kl} - \tfrac{1}{3}Q_{mm}\,\delta_{kl}, \tag{31}$$
$$Q_{kl} = \overline{u'_k\,\bar{u}_l} + \overline{\bar{u}_k\,u'_l} + \overline{u'_k\,u'_l}, \tag{32}$$
$$\bar{P} = \bar{p} + \tfrac{1}{3}Q_{mm}. \tag{33}$$

Since the correlation $\overline{\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}}}$ on the left-hand side of (29) is calculated directly, closure is achieved by simply taking $\boldsymbol{T}$ to be a functional of $\bar{\boldsymbol{u}}$. In Biringen & Reynolds (1981), the Smagorinsky (1963) model is used in the form
$$T_{kl} = -C_1\,\Delta^2 \Pi^{\frac{1}{2}}_{\bar{D}}\,\bar{D}_{kl}, \tag{34}$$
where
$$\bar{D}_{kl} = \frac{1}{2}\left(\frac{\partial \bar{u}_k}{\partial x_l} + \frac{\partial \bar{u}_l}{\partial x_k}\right), \tag{35}$$
$$\Pi_{\bar{D}} = \bar{D}_{mn}\,\bar{D}_{mn}, \tag{36}$$
and C_1 is a dimensionless constant. Similarly, in Moin & Kim (1982), the model
$$T_{kl} = -C_2\,\Delta^2 \Pi^{\frac{1}{2}}_{\bar{D}-\langle\bar{D}\rangle}(\bar{D}_{kl} - \langle\bar{D}_{kl}\rangle) - f\Delta^2\Pi^{\frac{1}{2}}_{\langle\bar{D}\rangle}\,\langle\bar{D}_{kl}\rangle \tag{37}$$
is chosen, where $\langle\cdot\rangle$ denotes an average over a plane parallel to the walls of the channel, C_2 is a dimensionless constant and f is a dimensionless wall damping function.

As a direct consequence of (3), (19) and (26), it is clear that
$$\bar{\boldsymbol{u}}^* = \bar{\boldsymbol{u}} + \boldsymbol{V}, \quad \boldsymbol{u}'^* = \boldsymbol{u}', \quad \overline{\boldsymbol{u}'^*} = \overline{\boldsymbol{u}'}, \tag{38}$$
$$\frac{\partial \bar{u}^*_k}{\partial x^*_l} = \frac{\partial \bar{u}_k}{\partial x_l}, \quad \bar{D}^*_{kl} = \bar{D}_{kl}, \quad \langle\bar{D}^*_{kl}\rangle = \langle\bar{D}_{kl}\rangle \tag{39}$$
under the Galilean group of transformations. Consequently, we have
$$\begin{aligned} Q^*_{kl} &= \overline{u'^*_k\,\bar{u}^*_l} + \overline{\bar{u}^*_k\,u'^*_l} + \overline{u'^*_k\,u'^*_l} \\ &= \overline{u'_k(\bar{u}_l + V_l)} + \overline{(\bar{u}_k + V_k)\,u'_l} + \overline{u'_k\,u'_l} \\ &= Q_{kl} + V_k\,\overline{u'_l} + V_l\,\overline{u'_k}, \end{aligned} \tag{40}$$
and hence the transformed version of the Biringen & Reynolds model (34) is given by
$$T^*_{kl} = V_k\,\overline{u'_l} + V_l\,\overline{u'_k} - \tfrac{2}{3}\boldsymbol{V}\cdot\overline{\boldsymbol{u}'}\,\delta_{kl} - C_1\,\Delta^2\Pi^{\frac{1}{2}}_{\bar{D}^*}\,\bar{D}^*_{kl}, \tag{41}$$
which is *not* form-invariant under a Galilean transformation. As a direct result of (39) and (40), it is also quite clear that the Moin & Kim model (37) is *not* Galilean-invariant! When (34) or (37) are substituted in (29), the resulting equations of motion for the large eddies that are obtained are also not Galilean-invariant. To be more specific, under a Galilean transformation the modelled versions of (29) and (30) take the forms
$$\frac{\partial \bar{u}^*_k}{\partial t^*} + \overline{\bar{u}^*_l\frac{\partial \bar{u}^*_k}{\partial x^*_l}} + V_l\frac{\partial \overline{u'^*_k}}{\partial x^*_l} = -\frac{\partial \bar{P}^*}{\partial x^*_k} + \nu\,\nabla^{*2}\bar{u}^*_k - \frac{\partial T^{\mathrm{M}*}_{kl}}{\partial x^*_l}, \tag{42}$$
$$\boldsymbol{\nabla}^*\cdot\bar{\boldsymbol{u}}^* = 0, \tag{43}$$
which is not invariant, since it depends explicitly on the translational velocity $\boldsymbol{V}$ of the frame. In (42)
$$\bar{P}^* = \bar{P} + \tfrac{2}{3}\boldsymbol{V}\cdot\overline{\boldsymbol{u}'} \tag{44}$$
and $\boldsymbol{T}^{\mathrm{M}*} = \boldsymbol{T}^{\mathrm{M}}$ is the modelled version of the subgrid-scale stress $\boldsymbol{T}$ given by the right-hand side of (34) or (37). It is thus clear that the equations of motion for the large eddies that are utilized in Biringen & Reynolds (1981) and Moin & Kim (1982)

are generally incompatible with the basic physics of the problem, which requires that the description of the turbulence be the same in all inertial frames of reference. This problem arises because the tensor $\boldsymbol{T}$, which is not Galilean-invariant, is replaced with an invariant expression, thus destroying the correct transformation properties of (27).

In order to avoid this problem of obtaining equations of motion for the large eddies that are not Galilean-invariant, subgrid-scale stress models must be form-invariant under the Galilean group of transformations. Stated mathematically, closure models for $\boldsymbol{\tau}$ (or any part of $\boldsymbol{\tau}$) must transform as

$$\tau^*_{kl}(\boldsymbol{x},t) = \tau_{kl}[\bar{\boldsymbol{u}}^*(\boldsymbol{x}',t);\boldsymbol{x}], \quad \boldsymbol{x}' \in D \tag{45}$$

under the Galilean group of transformations and thus be of the same form in all inertial frames of reference. By making use of (38), it is a simple matter to show that

$$L^*_{kl} = L_{kl} - V_k \overline{u'_l} - V_l \overline{u'_k}, \tag{46}$$

$$C^*_{kl} = C_{kl} + V_k \overline{u'_l} + V_l \overline{u'_k}, \tag{47}$$

$$R^*_{kl} = R_{kl}, \tag{48}$$

and hence

$$L^*_{kl} + C^*_{kl} = L_{kl} + C_{kl}, \tag{49}$$

$$\tau^*_{kl} = \tau_{kl} \tag{50}$$

under a Galilean transformation. Thus, while $\boldsymbol{\tau}, \boldsymbol{L}+\boldsymbol{C}$, and $\boldsymbol{R}$ are Galilean invariant, neither $\boldsymbol{L}$ nor $\boldsymbol{C}$ has this property by itself. In order to satisfy (45) in models where the Leonard stress $\boldsymbol{L}$ is calculated directly, the modelled version of $\boldsymbol{C}$ must have a Galilean variance identical with the last two terms on the right-hand side of (47). The Biringen & Reynolds (1981) model and the Moin & Kim (1982) model do not have this property, as is obvious from (41), and are thus physically inconsistent.

Now, it will be proven that the linear-combination model of Bardina *et al.* (1983) is Galilean-invariant provided that the Bardina constant is chosen appropriately. This model takes the form

$$C_{kl} = c_r(\bar{u}_k \bar{u}_l - \bar{\bar{u}}_k \bar{\bar{u}}_l), \tag{51}$$

$$R_{kl} - \tfrac{1}{3}R_{mm}\,\delta_{kl} = -C_1\,\Delta^2 \Pi^{\frac{1}{2}}_{\bar{D}}\,\bar{D}_{kl}, \tag{52}$$

where c_r is the dimensionless Bardina constant. Closure is achieved in this case by simply modelling $\boldsymbol{C}$ and $\boldsymbol{R}$, since the Leonard stress $\boldsymbol{L}$ is calculated directly. It is a simple matter to show that (51) and (52) transform as

$$C^*_{kl} - V_k \overline{u'_l} - V_l \overline{u'_k} = c_r(\bar{u}^*_k \bar{u}^*_l - \bar{\bar{u}}^*_k \bar{\bar{u}}^*_l) - c_r(V_k \overline{u'_l} + V_l \overline{u'_k}), \tag{53}$$

$$R^*_{kl} - \tfrac{1}{3}R^*_{mm}\,\delta_{kl} = -C_1\,\Delta^2 \Pi^{\frac{1}{2}}_{\bar{D}^*}\,\bar{D}^*_{kl} \tag{54}$$

under the Galilean group. Hence, if we take

$$c_r = 1,$$

it is clear that (53) reduces to the form

$$C^*_{kl} = \bar{u}^*_k \bar{u}^*_l - \bar{\bar{u}}^*_k \bar{\bar{u}}^*_l, \tag{55}$$

which is Galilean-invariant. It thus follows that the linear-combination model with $c_r = 1$ gives rise to equations of motion for the large eddies that are Galilean-invariant. Interestingly enough, Bardina *et al.* (1983) arrived at a value of $c_r = 1.1$ by correlating with data obtained from direct numerical simulations of homogeneous turbulence. However, in future calculations with the linear-combination model, c_r must be modified to a value of 1.

Finally, it will be demonstrated that the somewhat older models of Deardorff (1970) and Clark *et al.* (1979) are also Galilean-invariant. The Deardorff model is given by

$$L_{kl}+C_{kl} = 0, \tag{56}$$

$$R_{kl}-\tfrac{1}{3}R_{mm}\,\delta_{kl} = -C_1\,\Delta^2\Pi^{\frac{1}{2}}_{\bar{D}}\,\bar{D}_{kl}, \tag{57}$$

where (56) constitutes the Reynolds-averaging assumption. By utilizing (39), (48) and (49), it is a simple matter to show that the Galilean transformation of this model takes the form

$$L^*_{kl}+C^*_{kl} = 0, \tag{58}$$

$$R^*_{kl}-\tfrac{1}{3}R^*_{mm}\,\delta_{kl} = -C_1\,\Delta^2\Pi^{\frac{1}{2}}_{\bar{D}^*}\,\bar{D}^*_{kl}, \tag{59}$$

and is thus invariant. Hence the Deardorff model yields equations of motion for the large eddies that are Galilean-invariant. The Clark *et al.* (1979) model can be written in the form

$$L_{kl}+C_{kl} = \tfrac{1}{12}\Delta^2\frac{\partial\bar{u}_k}{\partial x_m}\frac{\partial\bar{u}_l}{\partial x_m}, \tag{60}$$

$$R_{kl}-\tfrac{1}{3}R_{mm}\,\delta_{kl} = -C_1\,\Delta^2\Pi^{\frac{1}{2}}_{\bar{D}}\,\bar{D}_{kl}, \tag{61}$$

where (60) is obtained by a Taylor expansion. By utilizing (39), (48) and (49), it is a simple matter to show that the Galilean transformation of this model is given by

$$L^*_{kl}+C^*_{kl} = \tfrac{1}{12}\Delta^2\frac{\partial\bar{u}^*_k}{\partial x^*_m}\frac{\partial\bar{u}^*_l}{\partial x^*_m}, \tag{62}$$

$$R^*_{kl}-\tfrac{1}{3}R^*_{mm}\,\delta_{kl} = -C_1\,\Delta^2\Pi^{\frac{1}{2}}_{\bar{D}^*}\,\bar{D}^*_{kl}, \tag{63}$$

which is form-invariant. The Clark *et al.* model thus gives rise to equations of motion for the large eddies which are also Galilean-invariant. It should be noted at this point that, while both the Deardorff model and the Clark *et al.* model are Galilean-invariant (and thus are physically consistent, unlike the Biringen & Reynolds (1979) and Moin & Kim (1982) approaches), they are inferior to the modified linear-combination model, since unnecessary errors are introduced in the modelling of the Leonard stresses. The Leonard stresses can be calculated directly, and this should be taken advantage of.

3. Conclusion

It has been proven that the models for the subgrid-scale stresses that are used in Biringen & Reynolds (1981) and Moin & Kim (1982) are not form-invariant under a Galilean transformation. Consequently, it follows that this approach gives rise to equations of motion for the evolution of the large eddies of turbulence that are not of the same form in all inertial frames of reference – a situation that is inconsistent with the basic physics of the problem. This difficulty arises because the subgrid-scale cross-stresses, which are not Galilean-invariant, are replaced through modelling with a Galilean-invariant term. In order to avoid this problem, any subgrid-scale stress models that are used must be form-invariant under a Galilean transformation. It was demonstrated that the linear-combination model of Bardina *et al.* (1983) is properly invariant provided that the Bardina constant is adjusted from a value of 1.1 to 1. It was also demonstrated that the older Deardorff (1970) model and the Clark *et al.* (1979) model are properly invariant. However, these models are not as good as the modified linear-combination model, since errors are unnecessarily introduced in the calculation of the Leonard stresses, as alluded to earlier.

From the standpoint of Galilean invariance and numerical accuracy in the calculation of the Leonard stresses, the modified linear-combination model appears to be the best existing subgrid-scale stress model. However, additional research is needed on the effects of rigid-body rotations and their dissipative structure on subgrid-scale stress models. This topic will be the subject of another paper.

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